Thermodynamic Bethe ansatz equation from fusion hierarchy of $osp(1|2)$ integrable spin chain

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Abstract

The thermodynamic Bethe ansatz (TBA) and the excited state TBA equations for an integrable spin chain related to the Lie superalgebra $osp(1|2)$ are proposed by the quantum transfer matrix (QTM) method. We introduce the fusion hierarchy of the QTM and derive the functional relations among them (T-system) and their certain combinations (Y-system). Their analytical property leads to the non-linear integral equations which describe the free energy and the correlation length at any finite temperatures. With regard to the free energy, they coincide with the TBA equation based on the string hypothesis.

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1 Introduction

Solvable lattice models related to the Lie superalgebras [1] have attracted a great deal of attention [2, 3, 4, 5, 6, 7, 8, 9]. For example, the supersymmetric $t-J$ model in strongly correlated electron system has received much attentions in relation with the high T_c superconductivity. These models have both fermionic and bosonic degree of freedom, and are given as

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solutions of the graded Yang-Baxter equations [2]. To solve such models, the Bethe ansatz is widely used in many literatures (see, for example, [3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and references therein). However, many of them deal only with models related to simple representations like fundamental ones; there was few *systematic* study by the Bethe ansatz on more complicated models such as fusion models [21].

In view of such situations, we have recently carried out [22, 23, 24, 25, 26] *systematically* an analytic Bethe ansatz [27, 28, 29, 30, 31, 32, 33] related to the Lie superalgebras $sl(r+1|s+1), B(r|s), C(s)$ and $D(r|s)$. Namely, we have proposed a class of dressed vacuum forms (DVFs) labeled by Young (super) diagrams and a set of fusion relations (T-system) among them.

Besides the eigenvalue formulae of the transfer matrices, the thermodynamics have also been discussed by several people. Particularly, the thermodynamic Bethe ansatz (TBA) [34] equation was proposed for the supersymmetric $t-J$ model [10], which is related to $sl(1|2)$, and the supersymmetric extended Hubbard model [14], which is related to $sl(2|2)$. Moreover, there is a paper [35] on the excited state TBA equation for these two important models of 1D highly correlated electron systems from the point of view of the quantum transfer matrix (QTM) method. In addition, the TBA equation for $sl(r|s)$ model was presented [36] in relation with the continuum limit of the integrable super spin chains.

However, the thermodynamics of the quantum spin model related to the orthosymplectic Lie superalgebra $osp(r|2s)$ is not so understood as $sl(r|s)$ case. In particular, as far as we know, there have been no literatures on the TBA equation even for the simplest orthosymplectic $osp(1|2)$ integrable spin chain [3, 4, 5, 7, 8, 9, 16, 17]. This is regrettable because this model may be related to interesting topics such as $N=1$ superconformal-symmetry in field theory, and the loop model, which will describe statistical properties of polymers in condensed matter physics [37]. In view of these situation, we have recently proposed the TBA equation for the $osp(1|2)$ integrable spin chain [38] by using the string hypothesis [39, 40, 41].

Though we expect that the resultant TBA equation describes the free energy correctly, there exists the deviation from the string hypothesis [42, 43, 44]. In addition, it is difficult to evaluate other physical quantities such as correlation length.

As an alternative method which overcomes such difficulties, the QTM method has been proposed [46, 47, 48, 49, 50, 51]. Now we shall briefly sketch the QTM method. Utilizing the general equivalence theorem [45, 46], one transforms the 1D quantum system into the 2D classical counterpart and defines the QTM on such a fictitious system of size N (referred to as the Trotter number, which should be taken $N \to \infty$). Since the QTM has a finite

gap, the original problem for the calculation of the partition function reduces to finding the single largest eigenvalue of the QTM. To evaluate it actually, we utilize the underlying integrable structure, which admits introduction of the "commuting" QTM with a complex parameter v [52, 53, 54, 55, 56, 57]. Furthermore, we introduce some auxiliary functions including the QTM itself, which satisfy functional relations.

We select these auxiliary functions such that they are *A*nalytic, *N*on*Z*ero and have *C*onstant asymptotics in appropriate strips on the complex v-plane (we call this property ANZC). Thus we can transform the functional relations into the non-linear integral equations (NLIE) which describe the free energy. In these NLIE, we can take the Trotter limit $N \to \infty$.

Adopting a subset of fusion hierarchy as auxiliary functions, we find that these NLIE are equivalent to the TBA equation based on the string hypothesis [35, 52, 58, 59, 60]. In general, a set of fusion hierarchy satisfies the functional relations called T-system, and this T-system is transformed to the Y-system [a]. By selecting them so that they have ANZC property in appropriate strips, one derives the NLIE which will be identical to the TBA equation. Furthermore, considering the sub-leading eigenvalues of the QTM, we can derive systematically the "excited state" TBA equations which provide the correlation length at any finite temperatures.

The purpose of this paper is to apply our recent results [26] to the $osp(1|2)$ integrable spin chain, and to construct the TBA equation and it's excited state version from the point of view of the above-mentioned QTM method. We have also confirmed the fact that our TBA equation coincides with the one [38] from the string hypothesis. We believe that this paper yields a basis of future studies of the thermodynamics by the QTM method for more general models such as the $osp(r|2s)$ model.

The layout of this paper is as follows. In section 2 we formulate the $osp(1|2)$ integrable spin chain at finite temperatures in terms of the commuting QTM. In section 3, to evaluate its eigenvalue, we utilize the fusion hierarchy of the QTM. It is given as a set of the dressed vacuum forms (DVFs) which is a summation over tableaux labeled by a Young (super) diagram with one column. These DVFs and their certain combinations (Y-functions) satisfy the T-system and the Y-system, respectively. The formulation of the DVFs and the functional relations are essentially independent of the vacuum part. Thus we can utilize the results in Ref. [26] only by replacing the vacuum parts of the DVFs defined in Ref. [26] with those of the QTM. Based on the analytical property of the DVFs and the Y-functions, in section 4 we

[a] For simple Lie algebras, general relations between T and Y-system are given in [61] and [62] (see also [63]).

derive the non-linear integral equations for the free energy, and in section 5 for the correlation length. Section 6 is devoted to the summary and the discussion.

2 Quantum transfer matrix

In this section, we introduce the quantum spin chain related to the Lie superalgebra $osp(1|2)$ and formulate the commuting QTM for this model.

The classical counterpart of this model is given by the $\check{R}$-matrix (see for example, Ref. [9])

$$\check{R}(v) = I + vP^g - \frac{v}{v-\frac{3}{2}}E, \tag{2.1}$$

which is a rational solution of the graded Yang-Baxter equation [2] associated with the three dimensional representation of the Lie superalgebra $osp(1|2)$ (to be precise, "super Yangian" $Y(osp(1|2))$), whose basis is $\mathbb{Z}_2$-graded and labeled by the parameters $p(1) = p(\overline{1}) = 1$, $p(0) = 0$. Here P^g denotes the graded permutation operator which has 9×9 matrix elements $(P^g)_{ab}^{cd} = (-1)^{p(a)p(b)}\delta_{a,d}\delta_{b,c}$ and E is a 9×9 matrix whose matrix elements are $E_{ab}^{cd} = \alpha_{ab}(\alpha^{-1})_{cd}$ where α is a 3×3 matrix

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{10} & \alpha_{1\overline{1}} \\ \alpha_{01} & \alpha_{00} & \alpha_{0\overline{1}} \\ \alpha_{\overline{1}1} & \alpha_{\overline{1}0} & \alpha_{\overline{1}\overline{1}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

and $\{a, b, c, d\} \in \{1, 0, \overline{1}\}$ with the total order $1 \prec 0 \prec \overline{1}$. The row-to-row transfer matrix $T(v)$ is defined by

$$T(v) = \mathrm{Tr}_a[R_{aL}(v) \cdots R_{a1}(v)], \tag{2.3}$$

where L is the number of lattice sites and $R_{aj}(v)$ denotes

$$R(v) = P\check{R}(v), \tag{2.4}$$

which acts non-trivially on the auxiliary space a and the j-th site of the quantum space. Note that $P_{ab}^{cd} = \delta_{a,d}\delta_{b,c}$ is the (non-graded) permutation operator. The Hamiltonian of the corresponding quantum system is given by taking the logarithmic derivative of above transfer matrix (2.3) at $v = 0$,

$$H = J\frac{d}{dv}\ln T(v)\Big|_{v=0} = J\sum_{j=1}^{L}\left(P_{j,j+1}^g + \frac{2}{3}E_{j,j+1}\right), \tag{2.5}$$

where we assume a periodic boundary condition. Here J is a real coupling constant which determines the phase of this model; the ferromagnetic and antiferromagnetic regimes correspond $J > 0$ and $J < 0$, respectively (see for example, [17, 38]).

To consider the finite temperature property of the model (2.5), we shall introduce another transfer matrix $\widetilde{T}(v)$ constructed by the R-matrix $\widetilde{R}(v)$, which is defined by 90° rotation of $R(v)$, i.e., $\widetilde{R}_{jk}(v) = {}^{t_k}R_{kj}(v)$ (t_k means the transposition for the R-matrix in the k-th space):

$$\widetilde{T}(v) = \mathrm{Tr}_a[\widetilde{R}_{aL}(v) \cdots \widetilde{R}_{a1}(v)]. \tag{2.6}$$

We can see that the logarithmic derivative of above transfer matrix (2.6) also represents the Hamiltonian (2.5). Thus the expansion of the transfer matrices (2.3) and (2.6) are expressed as

$$\begin{aligned} T(v) &= T(0)\left\{1 + \frac{H}{J}v + \mathcal{O}(v^2)\right\}, \\ \widetilde{T}(v) &= \widetilde{T}(0)\left\{1 + \frac{H}{J}v + \mathcal{O}(v^2)\right\}. \end{aligned} \tag{2.7}$$

Combining above relations and using the fact $T(0)\widetilde{T}(0) = 1$ (note that $T(0)$ and $\widetilde{T}(0)$ denote the right and left sift operators, respectively), we have

$$T(v)\widetilde{T}(v) = 1 + \frac{2H}{J}v + \mathcal{O}(v^2). \tag{2.8}$$

Consequently, the partition function Z of the model (2.5) can be expressed as

$$Z = \mathrm{Tr}e^{-\beta H} = \lim_{N\to\infty} \mathrm{Tr}\left(T(u_N)\widetilde{T}(u_N)\right)^{N/2}, \ u_N = -\frac{J\beta}{N}, \tag{2.9}$$

where N is an even number: it is called the Trotter number, which represents the number of fictitious sites in the Trotter direction; $\beta = 1/(k_B T)$ (T is the temperature). Thus the free energy per site f is given by

$$f = -\lim_{L\to\infty}\lim_{N\to\infty} \frac{1}{L\beta} \ln \mathrm{Tr}\left(T(u_N)\widetilde{T}(u_N)\right)^{\frac{N}{2}}. \tag{2.10}$$

However, in the antiferromagnetic case $J < 0$, the eigenvalues of $T(u_N)\widetilde{T}(u_N)$ will be infinitely degenerate in the limit $N \to \infty$. Therefore taking the trace in above expression (2.10) is a serious problem. To avoid this difficulty, we transform the term $T(u_N)\widetilde{T}(u_N)$ as follows:

$$\mathrm{Tr}\left(T(u_N)\widetilde{T}(u_N)\right)^{\frac{N}{2}} = \mathrm{Tr}\prod_{k=1}^{N/2}\mathrm{Tr}_{a_{2k},a_{2k-1}}\Big(R_{a_{2k},L}(u_N)\cdots R_{a_{2k},1}(u_N)$$

$$\begin{aligned} & \times \widetilde{R}_{a_{2k-1},L}(u_N) \cdots \widetilde{R}_{a_{2k-1},1}(u_N)\Big) \\ = \ & \mathrm{Tr} \prod_{j=1}^{L} \mathrm{Tr}_j \prod_{k=1}^{N/2} R_{a_{2k},j}(u_N) \widetilde{R}_{a_{2k-1},j}(u_N). \qquad (2.11) \end{aligned}$$

Now we introduce the QTM, which plays the fundamental role to describe the thermal quantities

$$T_1^{(1)}(u,v) = \mathrm{Tr}_j \prod_{m=1}^{N/2} R_{a_{2m},j}(u-iv) \widetilde{R}_{a_{2m-1},j}(u+iv). \qquad (2.12)$$

Due to the Yang-Baxter equation, we see that the QTM is commutative:

$$[T_1^{(1)}(u,v), T_1^{(1)}(u,v')] = 0. \qquad (2.13)$$

Hereafter we write the k-th largest eigenvalue of the QTM $T_1^{(1)}(u,0)$ as $T_{1,k}^{(1)}(u,0)$. Since the two limits in (2.10) will be exchangeable as proved in [46, 47], we shall take the limit $L \to \infty$ first. Since there is a finite gap between $T_{1,1}^{(1)}(u_N,0)$ and $T_{1,2}^{(1)}(u_N,0)$ even in the Trotter limit $N \to \infty$, we have

$$f = -\frac{1}{\beta} \lim_{N\to\infty} \ln T_{1,1}^{(1)}(u_N,0). \qquad (2.14)$$

The thermodynamical completeness $-\lim_{\beta\to 0} \beta f = \ln 3$ follows from $T_{1,1}^{(1)}(0,0) = 3$, which is clear from $R_{12}(0) = P_{12}$.

Taking the ratio of the largest eigenvalue and the second largest one, we can systematically calculate the correlation length ξ:

$$\frac{1}{\xi} = -\lim_{N\to\infty} \ln \left| \frac{T_{1,2}^{(1)}(u_N,0)}{T_{1,1}^{(1)}(u_N,0)} \right|. \qquad (2.15)$$

3 Fusion hierarchy and functional relations

In this section, we present the eigenvalue formulae of the fusion QTM $T_m^{(1)}(u,v)$ [b] and the functional relations among them by using the results for the row-to-row transfer matrix [26]. Although the quantity we want to evaluate is only $T_1^{(1)}(u,0)$, consideration on all $\{T_m^{(1)}(u,v)\}$ plays an essential role in our

[b] We also use the expression $T_m^{(1)}(u,v)$ as the eigenvalues of the fusion QTM.

formulation. For $a \in \{1, 0, \overline{1}\}$, we define the function $\boxed{a}_v$ as

$$
\begin{aligned}
\boxed{1}_v &= \psi_1(v)\frac{Q(v-\frac{i}{2})}{Q(v+\frac{i}{2})}, \\
\boxed{0}_v &= \psi_0(v)\frac{Q(v)Q(v+\frac{3i}{2})}{Q(v+\frac{i}{2})Q(v+i)}, \\
\boxed{\overline{1}}_v &= \psi_{\overline{1}}(v)\frac{Q(v+2i)}{Q(v+i)},
\end{aligned}
\tag{3.1}
$$

where $Q(v) = \prod_{j=1}^{n}(v - v_j)$; $n \in \{0, 1, \ldots, N\}$ is a quantum number; $v_j \in \mathbb{C}$. The vacuum parts of the functions $\boxed{a}_v$ (3.1) are given as follows:

$$
\begin{aligned}
\psi_1(v) &= (-1)^{N-n}\frac{\phi_+(v)\phi_-(v+i)\phi_+(v-\frac{i}{2})}{\phi_+(v-\frac{3i}{2})}, \\
\psi_0(v) &= \phi_+(v)\phi_-(v), \\
\psi_{\overline{1}}(v) &= (-1)^{N-n}\frac{\phi_-(v)\phi_+(v-i)\phi_-(v+\frac{i}{2})}{\phi_-(v+\frac{3i}{2})},
\end{aligned}
\tag{3.2}
$$

where

$$
\phi_\pm(v) = (v \pm iu)^{\frac{N}{2}}. \tag{3.3}
$$

The eigenvalue formula of the QTM is given as the summation over these boxes (3.1):

$$
T_1^{(1)}(u, v) = \boxed{1}_v + \boxed{0}_v + \boxed{\overline{1}}_v. \tag{3.4}
$$

We have [c]

$$
\begin{aligned}
&Res_{v=-\frac{i}{2}+v_k}(\boxed{1}_v + \boxed{0}_v) = 0, \\
&Res_{v=-i+v_k}(\boxed{0}_v + \boxed{\overline{1}}_v) = 0,
\end{aligned}
\tag{3.5}
$$

under the following type of the Bethe ansatz equation (BAE):

$$
\frac{\phi_-(v_k+\frac{i}{2})\phi_+(v_k-i)}{\phi_-(v_k-\frac{i}{2})\phi_+(v_k-2i)} = -(-1)^{N-n}\frac{Q(v_k-\frac{i}{2})Q(v_k+i)}{Q(v_k+\frac{i}{2})Q(v_k-i)} \quad \text{for} \quad k \in \{1, 2, \ldots, n\}. \tag{3.6}
$$

[c] Here $Res_{v=a}f(v)$ denotes the residue of a function $f(v)$ at $v = a$.

Thus $T_1^{(1)}(u,v)$ (3.4) is free of poles[d] under this BAE (3.6). We observe, from numerical analysis, that the largest and the second largest eigenvalues lie in the sector $n = N$ and $n = N-1$, respectively.

We shall introduce a function $T_m^{(1)}(u,v)$ with a spectral parameter $v \in \mathbb{C}$ and a Young (super) diagram [e] (1^m), which is a candidate of the DVF for the fusion QTM. From now on, we often abbreviate the parameter u to simplify the notation. For $m \in \mathbb{Z}_{\geq 1}$, we define $T_m^{(1)}(v)$ as a summation over products of the boxes in (3.1) :

$$T_m^{(1)}(v) \quad = \quad \frac{1}{\mathcal{N}_m(v)} \sum_{\{i_k\} \in B(1^m)} \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_m \\ \hline \end{array}, \tag{3.7}$$

where the spectral parameter v is shifted as $v + \frac{m-1}{2i}, v + \frac{m-3}{2i}, \ldots, v + \frac{-m+1}{2i}$ from the top to the bottom. $B(1^m)$ is a set of tableaux $\{i_k\}$ ($i_k \in \{1, 0, \overline{1}\}$) obeying the following rule (admissibility conditions)

$$i_1 \preceq i_2 \preceq \cdots \preceq i_m. \tag{3.8}$$

The normalization function $\mathcal{N}_m(v)$ is introduced so that the degree of $T_m^{(1)}(v)$ is $2N$ with respect to v:

$$\mathcal{N}_m(v) = \prod_{j=1}^{m-1} \phi_- \left(v + \frac{m+1-2j}{2} i \right) \phi_+ \left(v - \frac{m+1-2j}{2} i \right). \tag{3.9}$$

The DVF (3.7) is free of poles under the BAE (3.6) for any $m \in \mathbb{Z}_{\geq 1}$. Note that $T_m^{(1)}(v)$ (3.7) for $m = 1$ reduces to (3.4) since $\mathcal{N}_1(v) = 1$. The BAE (3.6) for the QTM has the same form as the one [3, 16, 17, 20] for the row to row transfer matrix except the vacuum part. Thus the dress part of the function $T_m^{(1)}(v)$ (3.7) is same as that in [26]. Therefore the DVFs $T_m^{(1)}(v)$ ($m \in \mathbb{Z}_{\geq 1}$) satisfy the following functional relations [f] ($osp(1|2)$ version of the fusion relations or the T-system [g]), which has essentially the same form in

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[d] Here singularities of the vacuum parts of the DVFs, which can be removed by multiplying overall vacuum functions are out of the question.

[e] Note that this diagram corresponds to the Kac-Dynkin label $2m$.

[f] Of course, there is a large class of functional relations among DVFs labeled by any skew-Young diagrams. The functional relations, which we are now dealing with is nothing but a subset of it. However we know, from our experience, that the functional relations among DVFs labeled by *rectangular* Young diagram often have a good analyticity relevant for the TBA. The T-system (3.10) is one of them.

[g] In this paper, we use the variable m in (3.10) slightly different from that in [26] for practical reason: the variable m in (3.10) corresponds to $2m$ in [26].

[26]:

$$T_m^{(1)}\left(v-\frac{i}{2}\right)T_m^{(1)}\left(v+\frac{i}{2}\right)=T_{m-1}^{(1)}(v)T_{m+1}^{(1)}(v)+T_m^{(0)}(v)T_m^{(1)}(v), \quad (3.10)$$

where

$$T_0^{(1)}(v)=\phi_-\left(v+\frac{i}{2}\right)\phi_+\left(v-\frac{i}{2}\right),$$

$$T_m^{(0)}(v)=\frac{\phi_-(v+\frac{m+2}{2}i)\phi_+(v-\frac{m+2}{2}i)\phi_-(v-\frac{mi}{2})\phi_+(v+\frac{mi}{2})}{\phi_-(v+\frac{m+1}{2}i)\phi_+(v-\frac{m+1}{2}i)}$$
$$\text{for} \quad m\in\mathbb{Z}_{\geq 1}. \quad (3.11)$$

We can prove these functional relations by the Jacobi identity and a duality among the DVFs [26].

Using $T_m^{(1)}(v)$ (3.7), we introduce the following functions (Y-functions):

$$Y_m(v):=\frac{T_{m-1}^{(1)}(v)T_{m+1}^{(1)}(v)}{T_m^{(0)}(v)T_m^{(1)}(v)} \quad \text{for} \quad m\in\mathbb{Z}_{\geq 1}. \quad (3.12)$$

These functions (3.12) satisfy a so-called Y-system:

$$Y_m(v-\frac{i}{2})Y_m(v+\frac{i}{2})=\frac{(1+Y_{m-1}(v))(1+Y_{m+1}(v))}{1+\{Y_m(v)\}^{-1}} \quad \text{for} \quad m\in\mathbb{Z}_{\geq 1}, (3.13)$$

where $T_{-1}^{(1)}(v)=Y_0(v)=0$. In the subsequent sections, these functional relations (3.13) will be transformed into the TBA equations.

4 Thermodynamic Bethe ansatz equation

Let $\{T_{m,k}^{(1)}(u,v)\}$ be the DVFs $\{T_m^{(1)}(u,v)\}$ constructed by the roots of BAE (3.6) which provide the k-th largest eigenvalue of the QTM $T_1^{(1)}(u,v)$, and $\{Y_{m,k}(u,v)\}$ be the Y-functions constructed by $\{T_{m,k}^{(1)}(u,v)\}$ as in (3.12).

In this section, we study the analyticity of the DVFs $\{T_{m,1}^{(1)}(u,v)\}$ and the Y-functions $\{Y_{m,1}(u,v)\}$ in the complex v-plane. Then we derive the non-linear integral equations (NLIE) which provide the free energy. For this purpose, by keeping the Trotter number N finite, we have performed numerical analyses with various values of β and N in investigating the location of zeros of $\{T_{m,1}^{(1)}(u,v)\}$. As seen in Fig.1, there are $N/2$ two-string solutions of the BAE (3.6), which will provide the largest eigenvalue of the QTM. These

roots locate symmetrically with respect to the imaginary axis and the line $\Im v = 3/4$.

From the definition of the DVFs (3.7), they have $2N$ zeros on the complex v-plane. For example, the location of zeros of $T^{(1)}_{m,1}(u,v)$ for $u = 0.05$, $N = 12$ is plotted in Fig.2. According to them, we observe that $T^{(1)}_{m,1}(u,v)$ have N zeros on the smooth curve near the line $\Im v = \pm(m+1)/2$ and N zeros on the smooth curve near the line $\Im v = \pm(m/2+1)$. Thus we expect that the following conjecture is valid even in the limit $N \to \infty$.

Conjecture 4.1 *All the zeros of $T^{(1)}_{m,1}(u,v)$ for small $|u| \ll 1$ are located outside of the strip $\Im v \in [-1/2, 1/2]$.*

From the definition (3.7), one observes the fact that all the zeros are symmetric with respect to both real and imaginary axes. As seen in the figure, the deviation from the line is very small, which will be smaller as $u \to 0$.

Once this conjecture is assumed, we can identify the strip where the Y-functions $\{Y_{m,1}(u,v)\}$ have the property of *A*nalytic *N*on*Z*ero and *C*onstant asymptotics in the limit $v \to \infty$ (we call it ANZC property). Using the definition of the DVFs (3.7) and the Y-functions (3.12), we can obtain the asymptotic value of the Y-functions in the limit $|v| \to \infty$:

$$\lim_{|v|\to\infty} Y_{m,1}(v) = \frac{m(m+3)}{2}. \tag{4.1}$$

From conjecture 4.1 and above asymptotics (4.1), the functions $1+Y_{m,1}(u,v)$ and $1+\{Y_{m,1}(u,v)\}^{-1}$ have ANZC property in the strip $\Im v \in [-\varepsilon, \varepsilon]$ ($0 < \varepsilon \ll 1$). On the other hand, the functions $Y_{m,1}(u,v)$ have the ANZC property in the strip $\Im v \in [-1/2, 1/2]$ (we call this strip physical strip), except for $Y_{1,1}(u,v)$ possesses poles (resp. zeros) of order $N/2$ at $\pm(1/2+u)i$ (resp. $\pm(1/2-u)i$) in the physical strip for $J > 0$ (resp. $J < 0$). Note that $u = u_N$ is a small quantity given in (2.9). Using Cauchy's theorem, we can transform the Y-system (3.13) into the NLIE in the following way. Let us consider the Y-system (3.13) for $m \geq 2$. First, we take the logarithmic derivative and perform the Fourier transformation on both side of the Eq.(3.13). Second, by Cauchy's theorem, the Fourier integral for the logarithmic derivative of $Y_{m,1}(v)$ is represented by that of $1 + Y_{m\pm1,1}(v)$ and $1 + \{Y_{m,1}(v)\}^{-1}$. Finally, performing the inverse Fourier transformation and integrating over v, we obtain the desired NLIE. The integral constants are determined by the asymptotic value in (4.1). In the case $m = 1$ in (3.13), we have to modify $Y_{1,1}(v)$ to transform Y-system into the NLIE since $Y_{1,1}(v)$ have the poles (or zeros) in the physical strip.

$$\widetilde{Y}_{1,1}(v) = Y_{1,1}(v)\{\tanh\frac{\pi}{2}(v + i(1/2 \pm u)) \tanh\frac{\pi}{2}(v - i(1/2 \pm u))\}^{\pm N/2}, \tag{4.2}$$

where the $+$ and $-$ signs in front of u and $N/2$ should be chosen according as $J > 0$ and $J < 0$, respectively. According to the identity $\tanh\frac{\pi}{4}(v+i)\tanh\frac{\pi}{4}(v-i) = 1$, $Y_{1,1}(v \pm i/2)$ in the lhs of the Eq.(3.13) can be replaced by $\widetilde{Y}_{1,1}(v)$ in (4.2). Thus the Y-system (3.13) can be transformed to the NLIE in the similar way as mentioned above. Consequently the resultant NLIE are represented as

$$\begin{aligned}
\ln Y_{1,1}(v) &= \mp\frac{N}{2}\ln\left\{\tanh\frac{\pi}{2}\left(v+i(1/2\pm u)\right)\tanh\frac{\pi}{2}\left(v-i(1/2\pm u)\right)\right\} \\
&\quad +K*\ln(1+Y_{2,1})(v) - K*\ln(1+Y_{1,1}^{-1})(v), \\
\ln Y_{m,1}(v) &= K*\ln(1+Y_{m+1,1})(v) + K*\ln(1+Y_{m-1,1})(v) \\
&\quad -K*\ln(1+Y_{m,1}^{-1})(v) \qquad \text{for } m \in \mathbb{Z}_{\geq 2},
\end{aligned} \tag{4.3}$$

where $A * B(v)$ denotes the convolution

$$A * B(v) = \int_{-\infty}^{\infty} A(v-v')B(v')dv', \tag{4.4}$$

and

$$K(v) = \frac{1}{2\cosh\pi v}. \tag{4.5}$$

In above NLIE (4.3), the Trotter limit $N \to \infty$ can be calculated:

$$\begin{aligned}
&\mp\lim_{N\to\infty}\frac{N}{2}\ln\left\{\tanh\frac{\pi}{2}\left(v+i(1/2\pm u_N)\right)\tanh\frac{\pi}{2}\left(v-i(1/2\pm u_N)\right)\right\} \\
&= \frac{\pi\beta J}{\cosh\pi v}.
\end{aligned} \tag{4.6}$$

We thus arrive at the NLIE for $Y_{m,1}(v)$ which are independent of the Trotter number N.

$$\begin{aligned}
\ln Y_{1,1}(v) &= \frac{\pi\beta J}{\cosh\pi v} + K*\ln(1+Y_{2,1})(v) - K*\ln(1+Y_{1,1}^{-1})(v), \\
\ln Y_{m,1}(v) &= K*\ln(1+Y_{m+1,1})(v) + K*\ln(1+Y_{m-1,1})(v) \\
&\quad -K*\ln(1+Y_{m,1}^{-1})(v) \qquad \text{for } m \in \mathbb{Z}_{\geq 2},
\end{aligned} \tag{4.7}$$

Under the identification $Y_{m,1}(v) = \eta_m(v)$, above equations (4.7) are identical to the TBA equation (30) in [38], which was derived by the thermodynamic Bethe ansatz based on the string hypothesis. From above equations and the asymptotics (4.1), we can determine the Y-functions $Y_{m,1}(v)$ uniquely.

To obtain the free energy per site, we shall modify $T_{1,1}^{(1)}(v)$ as

$$\widetilde{T}_{1,1}^{(1)}(v) = \frac{T_{1,1}^{(1)}(v)}{\phi_+(v-i)\phi_-(v+i)}. \tag{4.8}$$

From the definition of the Y-function (3.12) and the T-system (3.10) $\widetilde{T}^{(1)}_{1,1}(v)$ satisfies the following functional relation.

$$\frac{\widetilde{T}^{(1)}_{1,1}(v+\frac{i}{2})\widetilde{T}^{(1)}_{1,1}(v-\frac{i}{2})}{\widetilde{T}^{(1)}_{1,1}(v)} = F(v)(1+Y_{1,1}(v)), \tag{4.9}$$

where

$$F(v) = \frac{\phi_+(v+\frac{i}{2})\phi_-(v-\frac{i}{2})}{\phi_+(v-\frac{i}{2})\phi_-(v+\frac{i}{2})}. \tag{4.10}$$

ANZC property of the both sides leads to

$$\begin{aligned}\ln T^{(1)}_{1,1}(u_N, v) &= G * \ln(1+Y_{1,1})(v) + \ln \phi_+(v-i)\phi_-(v+i) \\ &\quad + N \int_0^\infty \frac{2e^{-\frac{k}{2}} \sinh k u_N \cos kv}{k(2\cosh\frac{k}{2}-1)} dk, \end{aligned} \tag{4.11}$$

where $G(v)$ denotes the kernel,

$$G(v) = \frac{2}{\sqrt{3}} \frac{\sinh\frac{4}{3}\pi v}{\sinh 2\pi v}. \tag{4.12}$$

Calculating the Trotter limit $N \to \infty$, we obtain the free energy per site (2.14)

$$f = J\left(\frac{4\pi}{3\sqrt{3}} - 1\right) - k_B T \int_{-\infty}^{\infty} G(v) \ln(1+Y_{1,1}(v)) dv. \tag{4.13}$$

This representation coincides with Eq.(33) in [38].

5 Excited state TBA equation

In this section, we develop the analysis for the largest eigenvalue of the QTM to the the analysis of the second largest one. Namely we derive the NLIE (excited state TBA equation) which provide the correlation length at finite temperature. We assume that the coupling constant J is negative: we consider only the antiferromagnetic regime in this section. After some numerical analyses, we observe that $n = N-1$ roots $\{v_k\}$ of the BAE (3.6), which will bring us the second largest eigenvalue of the QTM, form $\frac{N}{2}-2$ two-strings and one three-string if $N \in 4\mathbb{Z}$ (cf. Fig. 3); $\frac{N}{2}-1$ two-strings and one one-string if $N \in 4\mathbb{Z}+2$ (cf. Fig. 5). As in the largest eigenvalue case, the distribution of these roots are symmetric with respect to the imaginary axis and the line $\Im v = 3/4$.

From the definition of the DVF (3.7), $T^{(1)}_{m,2}(u,v)$ has $2N$ zeros on the complex v-plain. From the numerical analyses, we observe that $T^{(1)}_{m,2}(u,v)$ has $N-2$ zeros on the smooth curve near the line $\Im v = \pm(m+1)/2$, $N-2$ zeros on the smooth curve near the line $\Im v = \pm(m/2+1)$, two zeros on the imaginary axis near the points $v = \pm(2m+3)i/4$ and two zeros $\pm x_m$ ($x_m > 0$) on the real axis. For example, the location of zeros of $T^{(1)}_{m,2}(v)$ for $u = 0.05$ and $N = 12, 14$ are plotted in Figs. 4,6. We expect that the following conjecture is valid even in the limit $N \to \infty$.

Conjecture 5.1 $T^{(1)}_{m,2}(u,v)$ *has two real zeros* $\pm x_m$. *Every other zero is located outside of the physical strip* $\Im v \in [-1/2, 1/2]$.

As in the previous section, we shall identify the strip where the Y-functions $\{Y_{m,2}(u,v)\}$ have the ANZC property. From the definition of the DVFs (3.7) and the Y-functions (3.12), we find the asymptotic value of $Y_{m,2}(v)$:

$$\lim_{|v|\to\infty} Y_{m,2}(v) = \frac{(-1)^m(2m+3)-3}{4}. \tag{5.1}$$

From the definition of (3.12) and conjecture 5.1, we find $(1+Y_{m+1,2}(v))(1+Y_{m-1,2}(v))/(1+\{Y_{m,2}(v)\}^{-1})$ has the ANZC property in the strip $\Im v \in [-\varepsilon, \varepsilon]$ ($0 < \varepsilon \ll 1$), while $Y_{m,2}(v)$ has zeros of order 1 at $v = \pm x_{m+1}, \pm x_{m-1}$ and poles of order 1 at $v = \pm x_m$ if $m \in \mathbb{Z}_{\geq 2}$; $Y_{1,2}(v)$ has zeros of order $N/2$ at $v = \pm(1/2-u)i$, zeros of order 1 at $v = \pm x_2$ and poles of order 1 at $v = \pm x_1$, and that every other singular point locates outside of the physical strip. Thus we have to modify $Y_{m,2}(v)$ as follows:

$$\begin{aligned}
\widetilde{Y}_{m,2}(v) \quad = \quad & Y_{m,2}(v)\left\{\tanh\frac{\pi}{2}(v+i(\frac{1}{2}-u))\tanh\frac{\pi}{2}(v-i(\frac{1}{2}-u))\right\}^{-\frac{N\delta_{m,1}}{2}} \\
& \times\left\{\tanh\frac{\pi}{2}(v-x_m)\tanh\frac{\pi}{2}(v+x_m)\right\} \\
& \times\left\{\tanh\frac{\pi}{2}(v-x_{m+1})\tanh\frac{\pi}{2}(v+x_{m+1})\right\}^{-1} \\
& \times\left\{\tanh\frac{\pi}{2}(v-x_{m-1})\tanh\frac{\pi}{2}(v+x_{m-1})\right\}^{-1+\delta_{m,1}}.
\end{aligned} \tag{5.2}$$

These $\widetilde{Y}_{m,2}(v)$ (5.2) have the ANZC property in the physical strip. After similar calculation in the previous section, we obtain the excited state TBA equation:

$$\ln Y_{1,2}(v) = \frac{\pi\beta J}{\cosh \pi v} + K * \ln\left\{\frac{1+Y_{2,2}}{1+Y_{1,2}^{-1}}\right\}(v)$$

$$
\begin{aligned}
&-\ln\left\{\tanh\frac{\pi}{2}(v-x_1)\tanh\frac{\pi}{2}(v+x_1)\right\}\\
&+\ln\left\{\tanh\frac{\pi}{2}(v-x_2)\tanh\frac{\pi}{2}(v+x_2)\right\}+\pi i,\\
\ln Y_{m,2}(v) &= K*\ln\left\{\frac{(1+Y_{m+1,2})(1+Y_{m-1,2})}{1+Y_{m,2}^{-1}}\right\}(v)\\
&-\ln\left\{\tanh\frac{\pi}{2}(v-x_m)\tanh\frac{\pi}{2}(v+x_m)\right\}\\
&+\ln\left\{\tanh\frac{\pi}{2}(v-x_{m+1})\tanh\frac{\pi}{2}(v+x_{m+1})\right\}\\
&+\ln\left\{\tanh\frac{\pi}{2}(v-x_{m-1})\tanh\frac{\pi}{2}(v+x_{m-1})\right\}+\frac{1+(-1)^{m+1}}{2}\pi i\\
&\text{for}\quad m\in\mathbb{Z}_{\geq 2},
\end{aligned}
\tag{5.3}
$$

where the integral kernel is defined in (4.5). In addition x_m is determined by the condition $Y_m(x_m \pm i/2) = -1$. Through the above excited state TBA equation (5.3), the second largest eigenvalue is given as follows:

$$
\begin{aligned}
\lim_{N\to\infty}\ln\left|T_{1,2}^{(1)}(u_N,0)\right| &= -\beta J\left(\frac{4\pi}{3\sqrt{3}}-1\right)+2\ln\left(\tanh\frac{\pi}{2}x_1\right)\\
&+\int_{-\infty}^{\infty}G(v)\ln\{(1+Y_{1,2}(v))\tanh\frac{\pi}{2}(v-x_1)\tanh\frac{\pi}{2}(v+x_1)\}dv.
\end{aligned}
\tag{5.4}
$$

This second largest eigenvalue (5.4) together with the largest one (4.13) describe the correlation length at any finite temperature (see, (2.15)). These excited state TBA equation (5.3) and Eq. (5.4) will be difficult to obtain by the string hypothesis; they exemplify the efficiency of the QTM method.

6 Summary and discussion

We have applied the QTM method to the $osp(1|2)$ integrable spin chain. Making use of the T-system and the Y-system, we derived the infinite set of NLIE which provide the free energy and the correlation length at any finite temperatures. As concerns the free energy, these NLIE are identical to the TBA equation based on the string hypothesis. One can say that the validity of the $osp(1|2)$ T-system in Ref. [26] has been confirmed from the point of view of the QTM method. To the author's knowledge, this paper is the first explicit derivation of the excited state TBA equation for the $osp(1|2)$ spin chain.

We comment on the ferromagnetic regime $J > 0$ in the second largest eigenvalue sector, which we have not considered in section 5. In the antiferromagnetic regime $J < 0$, the correlation length are characterized by

the additional zeros of the fusion QTMs. These zeros are real and symmetric with respect to the imaginary axis and this symmetry will never break at any finite temperature. While in the ferromagnetic regime ($J > 0$), the "level crossing" will occur successively. This attributes to the change of distribution patterns of the additional zeros as follows. First at high temperature, two additional zeros are on the imaginary axis in the physical strip. However, at low temperature, this pattern will no longer provide the second largest eigenvalue.

There is another definition of the transfer matrix for the R-matrix (2.4), which is based on the *graded* formalism [3, 2, 11, 20] of the quantum inverse scattering method. In this case, the R-matrix (2.4) is defined as $R(v) = P^g \check{R}(v)$, where P^g is the *graded* permutation operator. As a result, the ordinary row-to-row transfer matrix is defined as the *super*-trace of a monodromy matrix, and the phase factor (cf.[16, 17]) of the BAE disappears [3, 20]. In the QTM for the graded case, we are not sure how the phase factor corresponding to the right hand side in (3.6) changes.

One might suspect that our TBA equation (4.7) is related to the one [64] for the Izergin-Korepin model since the representation space of $osp(1|2)^{(1)}$ has close resemblance to that of the affine Lie algebra $A_2^{(2)}$. However, to our knowledge, the TBA equation for the Izergin-Korepin model has been constructed [64] so that it reduces to the one for a $su(3)$-invariant model in the rational limit. There will be another branch of the rational limit for the Izergin-Korepin model [17]. Together with the above-mentioned graded case, the comparison with our results will be interesting.

It is an interesting problem to extend a similar analysis to more general $osp(r|2s)$ model. Unfortunately, for $r \in \mathbb{Z}_{\geq 2}$ case, we have [h] only the subset of the T-system [26, 25]. While for $r = 1$ case, we have already a complete set of the T-system [26], which will be relevant for the QTM method. So we hope to report, at the beginning, the TBA equation for the $osp(1|2s)$ model in the near future.

Finally we refer to another formulation of NLIE based on the finite number of the auxiliary functions (see for example, [52, 53, 54, 55, 56, 57]). Although there seems to be no connection between the TBA equation and such NLIE, this formulation will be advantageous to the numerical calculation of the physical quantities at finite temperature.

[h] $osp(2|2) \simeq sl(1|2)$ case is an exception to this statement [22, 23, 24, 25, 35].

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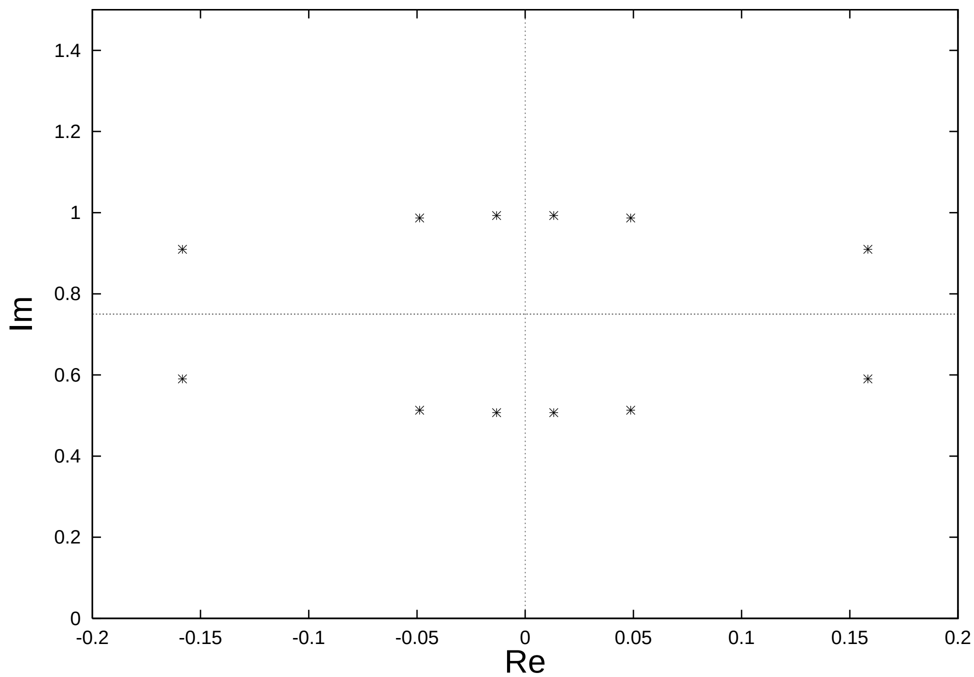

Figure 1: The distribution of the BAE roots which provides the largest eigenvalue for $N = 12$, $u = 0.05$. There are six two-string solutions which are symmetric with respect to the imaginary axis and the line $\Im v = 3/4$.

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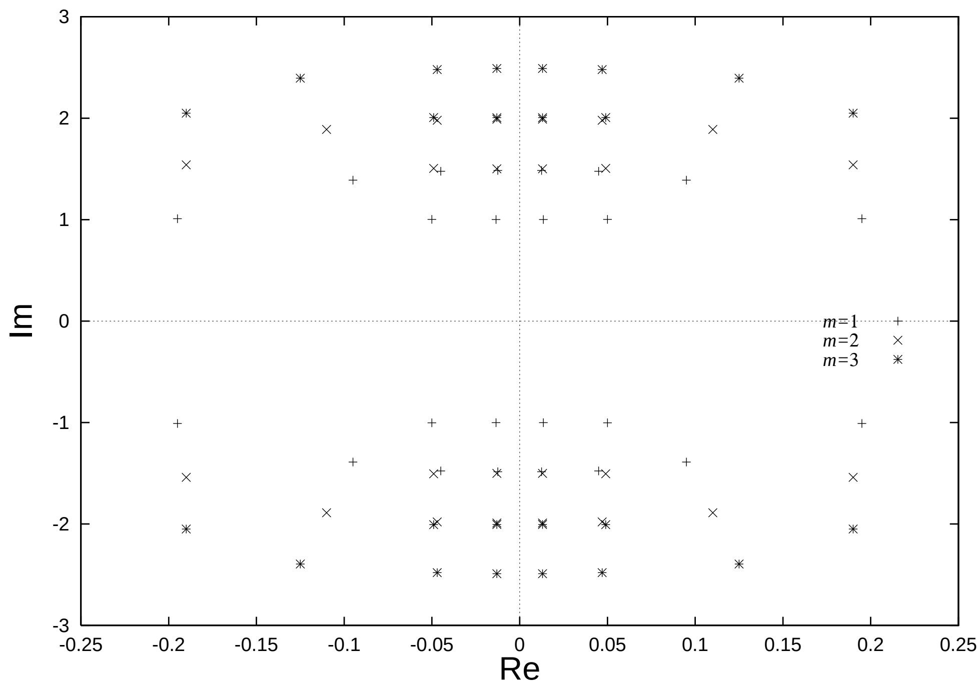

Figure 2: Location of zeros for $T^{(1)}_{m,1}(u,v)$ for $m = 1, 2, 3$, $u = 0.05$ and $N = 12$. Note that these zeros are distributed symmetrically with respect to the real and the imaginary axis.

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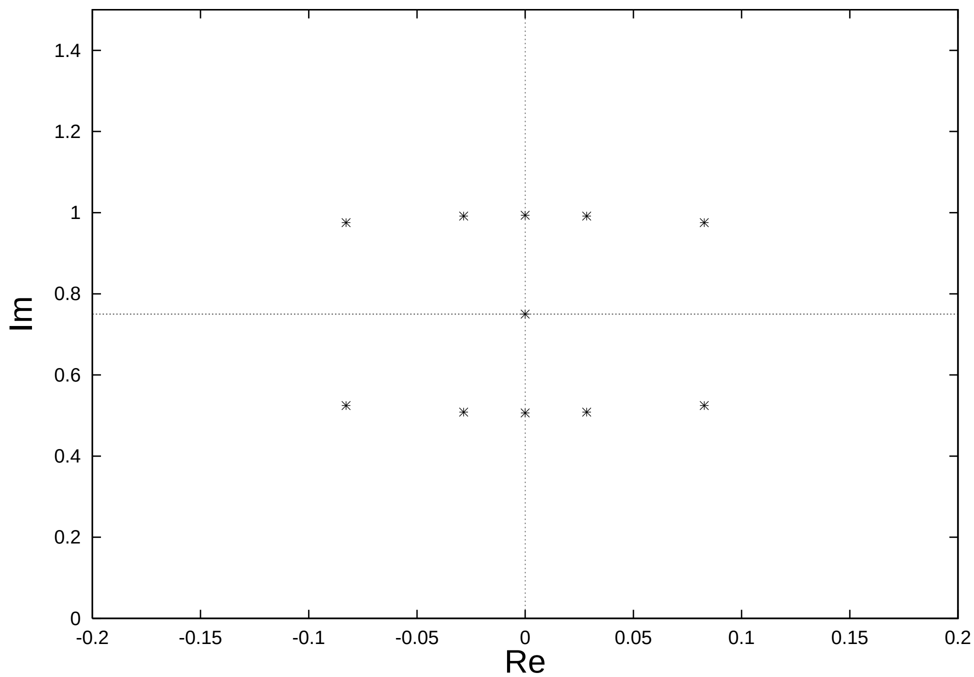

Figure 3: Location of the roots of the BAE (3.6) for $u = 0.05$, $N = 12$ and $n = 11$, which gives the second largest eigenvalue of the QTM. There are four two-strings and one three-string. They are symmetrically distributed with respect to the line $\Im v = 3/4$ and the imaginary axis.

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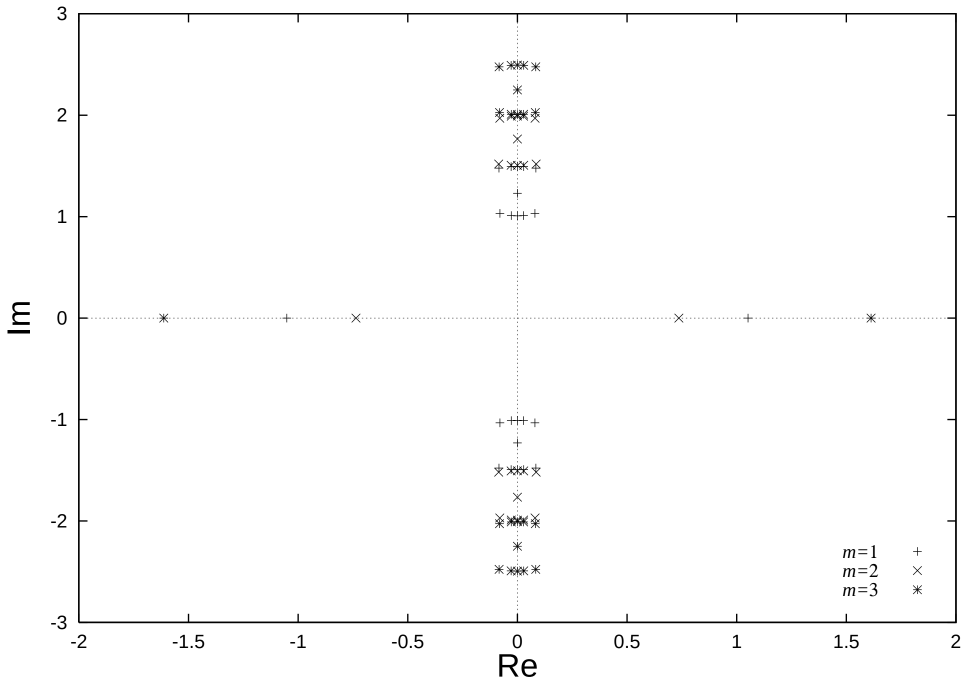

Figure 4: Location of zeros of $T^{(1)}_{m,2}(u, v)$ for $m = 1, 2, 3$, $u = 0.05$ and $N = 12$. Note that these zeros are distributed symmetrically with respect to the real and the imaginary axis.

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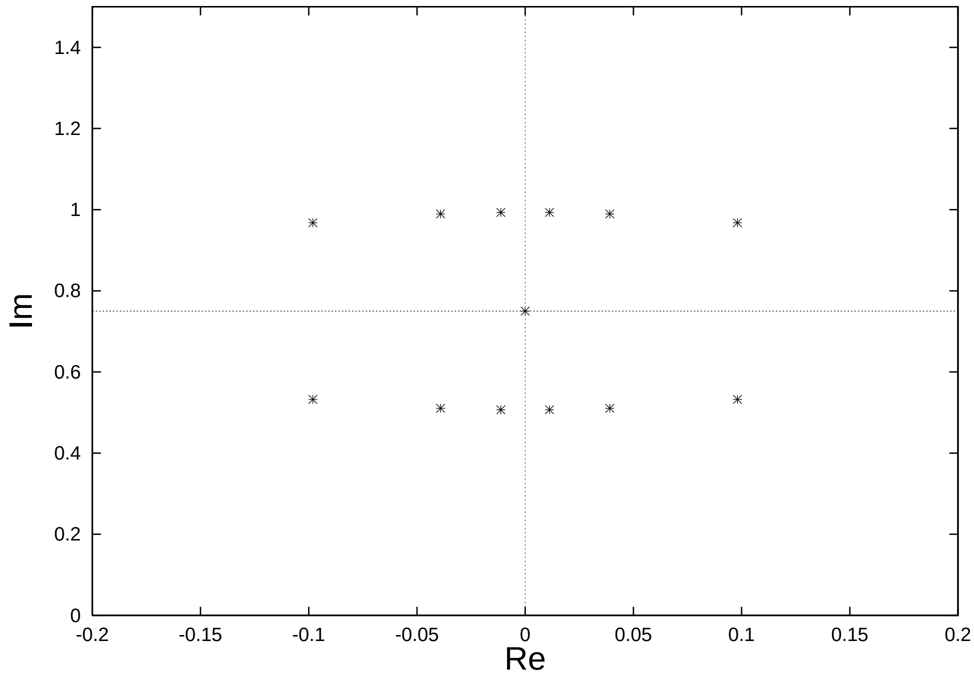

Figure 5: Location of the roots of the BAE (3.6) for $u = 0.05$, $N = 14$ and $n = 13$, which gives the second largest eigenvalue of the QTM. There are six two-strings and one one-string. They are symmetrically distributed with respect to the line $\Im v = 3/4$ and the imaginary axis.

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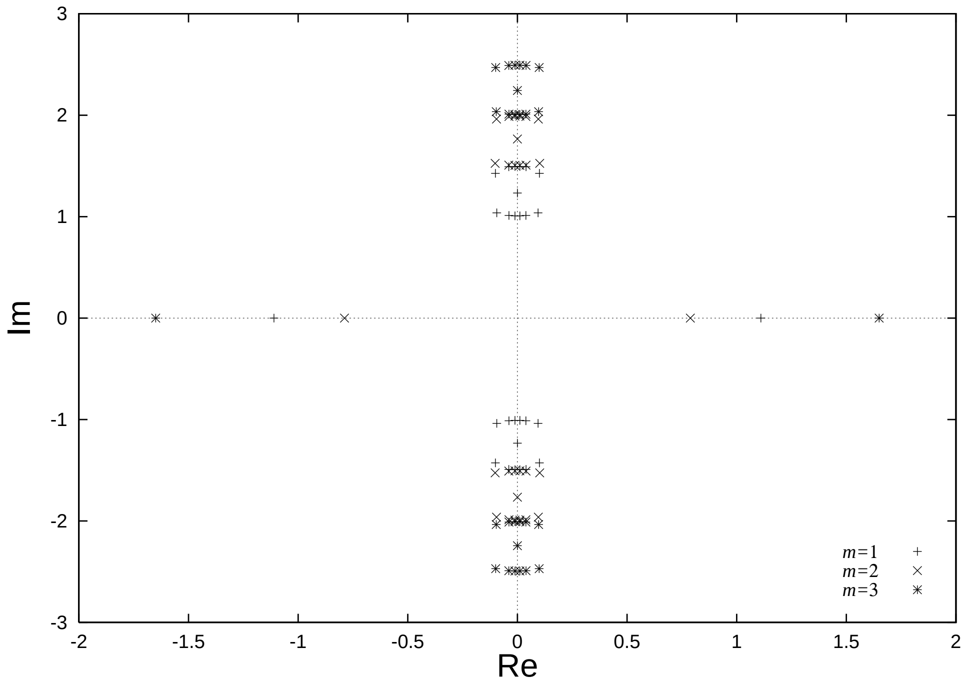

Figure 6: Location of zeros of $T^{(1)}_{m,2}(u,v)$ for $m=1,2,3$, $u=0.05$ and $N=14$. Note that these zeros are distributed symmetrically with respect to the real and the imaginary axis.

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